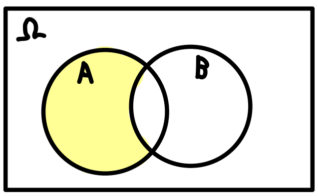
CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Koushik Sen and Satish Rao — DIS 8B

# 1 Probability Potpourri

Provide brief justification for each part.

(a) For two events $A$ and $B$ in any probability space, show that $\mathbb{P}[A \setminus B] \geq \mathbb{P}[A] - \mathbb{P}[B]$.




$$A = (A \setminus B) \cup (A \cap B)$$

$$P[A] = P[A \setminus B] + P[A \cap B]$$

$$P[A] - P[A \cap B] = P[A \setminus B]$$

Since $A \cap B \subseteq B$, $P[A \cap B] \leq P[B]$.

Thus $P[A \setminus B] \geq P[A] - P[B]$.

(b) Suppose $\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \overline{C}]$, where $\overline{C}$ is the complement of $C$. Prove that $D$ is independent of $C$.

$$\begin{aligned} P[D] &= P[D \mid C]\,P[C] + P[D \mid \bar{C}]\,P[\bar{C}] \\ &= P[D \mid C]\,P[C] + P[D \mid C]\,P[\bar{C}] \\ &= P[D \mid C]\left(P[C] + P[\bar{C}]\right) \\ &= P[D \mid C] \end{aligned}$$

(c) If $A$ and $B$ are disjoint, does that imply they're independent?

No. disjoint: $P[A \cap B] = 0$

independent: $P[A \cap B] = P[A]P[B]$

## 2 Aces

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

$P[\text{ace}] = \frac{4}{52}$

$P[\text{red card}] = \frac{26}{52}$

$P[\text{ace} \cap \text{red card}] = \frac{2}{52}$

$P[\text{ace} \cup \text{red card}] = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

$P[\text{ace} \cup \text{spade}] - P[\text{ace} \cap \text{spade}] = \left(\frac{4}{52} + \frac{13}{52} - \frac{1}{52}\right) - \frac{1}{52} = \frac{15}{52}$

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

$$\frac{\binom{1}{1}\binom{51}{4}}{\binom{52}{5}} = \frac{\binom{51}{4}}{\binom{52}{5}} = \frac{\frac{51!}{4!\,47!}}{\frac{52!}{5!\,47!}} = \frac{5}{52}$$

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

$$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$$

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

$$1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

$$1 - \frac{\binom{36}{5}}{\binom{52}{5}}$$

## 3 Balls and Bins

Throw $n$ balls into $n$ labeled bins one at a time.

(a) What is the probability that the first bin is empty?

$$\frac{(n-1)^n}{n^n} = \left(\frac{n-1}{n}\right)^n$$

(b) What is the probability that the first $k$ bins are empty?

$$\left(\frac{n-k}{n}\right)^n$$

(c) Let $A$ be the event that at least k bins are empty. Notice that there are $m$ subsets of $k$ bins out of the total $n$ bins. If we assume $A_i$ is the event that the $i^{\text{th}}$ set of $k$ bins is empty. Then we can write A as the union of $A_i$'s.

$$A = \bigcup_{i=1}^{m} A_i.$$

Use the union bound to give an upper bound on the probability $A$ from part (c).

$$P[A] \leq \sum_{i=1}^{m} P[A_i]$$

$$\leq \sum_{i=1}^{m} \left(\frac{n-k}{n}\right)^n$$

$$\leq m\left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n$$

(d) What is the probability that the second bin is empty given that the first one is empty?

$$\frac{\left(\frac{n-2}{n}\right)^n}{\left(\frac{n-1}{n}\right)^n} = \left(\frac{n-2}{n-1}\right)^n$$

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

No

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

No