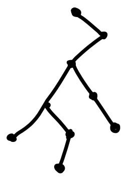
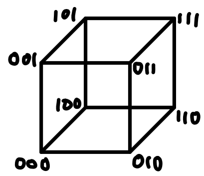


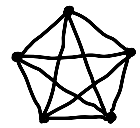
Tree

n vertices
$n-1$ edges



Hypercube

2^n vertices
$n2^{n-1}$ edges



Complete Graph

n vertices
$\frac{n(n-1)}{2}$ edges

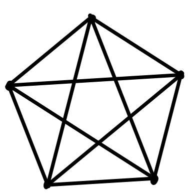
Planar Graphs

$v + f = e + 2$ (for connected planar graphs)

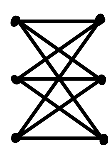
$3f \leq 2e$ (for $e \geq 2$)

$e \leq 3v - 6$ (for $v \geq 3$)

Nonplanar Graphs



K_5



$K_{3,3}$

CS 70 Discrete Mathematics and Probability Theory
Spring 2022 Koushik Sen and Satish Rao **DIS 2B**

1 Short Answers

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

a) $v + f = e + 2$

$v + f = (v+5) + 2$

$f = 7$

b) 3D Hypercube

8 vertices

12 edges

5 edges must be removed

2 Always, Sometimes, or Never

In each part below, you are given some information about the so-called original graph, *OG*. Using only the information in the current part, say whether *OG* will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) *OG* can be vertex-colored with 4 colors.

(b) *OG* requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where e is the number of edges of *OG* and v is the number of vertices of *OG*.

(d) *OG* is connected, and each vertex in *OG* has degree at most 2.

(e) Each vertex in *OG* has degree at most 2.

a) either, any planar graph and $K_{3,3}$ can be 4-colored

b) nonplanar, by 5-color theorem

c) either, $K_{3,3}$ satisfies this

d) planar, OG is tree or cycle

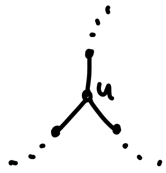
e) planar, OG cannot contain $K_{3,3}$ or K_5

3 Trees and Components

(a) Bob removed a degree 3 node from an n-vertex tree. How many connected components are there in the resulting graph? Please provide an explanation.

(b) Given an n-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph? Please provide an explanation.

a) 3



After removing u, the 3 neighboring vertices are all in different connected components.

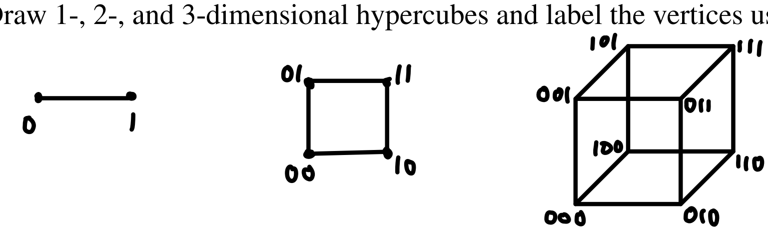
b) 7

Graph has $n+4$ edges. To create 3 connected components where each connected component is a tree, there must be $n-3$ edges.

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.



(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

$L = \{v \in V \mid v \text{ has odd \# of 1's in bitstring}\}$

$R = \{v \in V \mid v \text{ has even \# of 1's in bitstring}\}$