Proof By Induction

To prove a proposition $P(n)$ for all $n \in \mathbb{N}$ by induction, your proof needs 3 parts.

1. Base Case: Prove $P(0)$.
2. Induction Hypothesis: Assume $P(k)$ for some $k \geq 0$.
3. Inductive Step: Prove $P(k) \Rightarrow P(k+1)$.

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1 Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1+x)^n \geq 1+nx$.

Base Case: $(1+x)^0 = 1$ $\quad 1 \geq 1$
$1 + 0x = 1$

Induction Hypothesis: $(1+x)^k \geq 1+kx$ for some $k \in \mathbb{N}$

Inductive Step: $(1+x)^{k+1} = (1+x)(1+x)^k$
$\geq (1+x)(1+kx)$
$\geq 1 + kx + x + kx^2$
$\geq 1 + (k+1)x$

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

Inductive Step: $a_k \leq 3^{(2^k)} \Rightarrow a_{k+1} \leq 3^{(2^{k+1})}$

$a_{k+1} = 3a_k^2 \leq 3(3^{2^k})^2 = 3(3^{2^{k+1}})$

We cannot prove $a_{k+1} \leq 3^{(2^{k+1})}$.

(b) Try to instead prove the statement $a_n \leq 3^{(2^n - 1)}$ using induction.

Base Case: $a_1 = 1 \leq 3^{2^1 - 1} = 3$

Induction Hypothesis: $a_k \leq 3^{(2^k - 1)}$ for some $k \in \mathbb{N}$

Inductive Step: $a_{k+1} = 3a_k^2 \leq 3(3^{(2^k - 1)})^2 = 3(3^{2^{k+1} - 2}) = 3^{2^{k+1} - 1}$

(c) Why does the hypothesis in part (b) imply the overall claim?

$a_n \leq 3^{(2^n - 1)}$ is a stronger statement,

since if $a_n \leq 3^{(2^n - 1)}$ and $3^{(2^n - 1)} \leq 3^{(2^n)}$, then $a \leq 3^{(2^n)}$

by transitive property.

3 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_i \in \{0,1\}$ for all $i \leq k$.

Base Case: When $n = 0$, we have $n = 0 \cdot 2^0$.

Induction Hypothesis: Assume every $1 \leq m \leq n$ can be written in binary.

Inductive Step: Prove $n+1$ can be written in binary.

If $n+1$ is even, then multiply both sides of $\frac{n+1}{2} = c_k \cdot 2^k + \ldots + c_0 \cdot 2^0$ to get $n+1 = c_k \cdot 2^{k+1} + \ldots + c_0 \cdot 2^1 + 0 \cdot 2^0$.

If $n+1$ is odd, then let $n = c_k \cdot 2^k + \ldots + c_0 \cdot 2^0$. Since n is even, then c_0 must be 0, otherwise RHS is odd. Thus, we can let $n+1 = c_k \cdot 2^k + \ldots + c_1 \cdot 2^1 + 1 \cdot 2^0$.

4 Fibonacci for Home

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

Base Case: $F_3 = 2$ is even.

Induction Hypothesis: $F_{3k} = 2q$ for some $q \in \mathbb{Z}$.

Inductive Step:

$$\begin{aligned} F_{3k+3} &= F_{3k+1} + F_{3k+2} \\ &= F_{3k+1} + (F_{3k} + F_{3k+1}) \\ &= 2F_{3k+1} + F_{3k} \\ &= 2F_{3k+1} + 2q \\ &= 2(F_{3k+1} + q) \end{aligned}$$