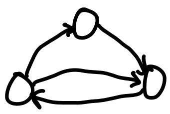
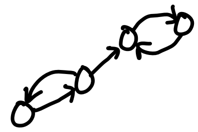
Markov Chains



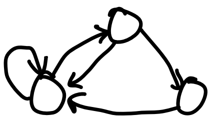
irreducible

It is possible to go from any state to any other state.



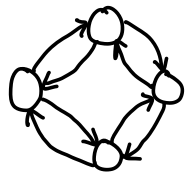
reducible

It is possible to get "stuck" somewhere!



aperiodic

The Markov chain is not periodic.



periodic

We always oscillate between certain states!

1 Markov Chain Basics

A Markov chain is a sequence of random variables X_n, $n = 0,1,2,\ldots$. Here is one interpretation of a Markov chain: X_n is the state of a particle at time n. At each time step, the particle can jump to another state. Formally, a Markov chain satisfies the Markov property:

$$\mathbb{P}[X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] = \mathbb{P}[X_{n+1} = j \mid X_n = i], \tag{1}$$

for all n, and for all sequences of states $i_0, \ldots, i_{n-1}, i, j$. In other words, the Markov chain does not have any memory; the transition probability only depends on the current state, and not the history of states that have been visited in the past.

(a) In lecture, we learned that we can specify Markov chains by providing three ingredients: $\mathscr{X}$, P, and π_0. What do these represent, and what properties must they satisfy?

X: set of state

P: transition probabilities

π_0: initial distribution

$P(i,j)$ = probability of transitioning from state i to state j

(b) If we specify $\mathscr{X}$, P, and π_0, we are implicitly defining a sequence of random variables X_n, $n = 0,1,2,\ldots$, that satisfies (1). Explain why this is true.

$P[X_0 = i] = \pi_0(i)$

$P[X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] = P[X_{n+1} = j \mid X_n = i] = P(i,j)$

X, P, and π_0 perfectly describe the distributions of X_n.

(c) Calculate $\mathbb{P}[X_1 = j]$ in terms of π_0 and P. Then, express your answer in matrix notation. What is the formula for $\mathbb{P}[X_n = j]$ in matrix form?

$$P[X_1 = j] = \sum_{i \in X} P[X_0 = i]\, P[X_1 = j \mid X_0 = i]$$

$$= \sum_{i \in X} \pi_0(i)\, P(i,j)$$

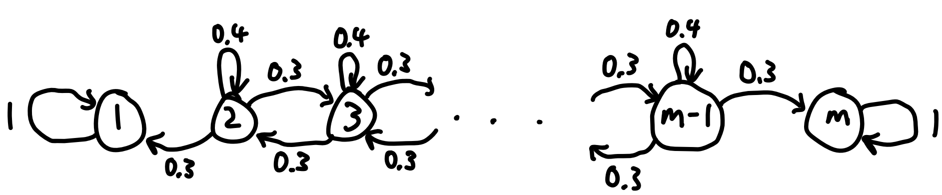
$$\pi_0 = \begin{bmatrix} \pi_0(1) & \pi_0(2) & \cdots & \pi_0(m) \end{bmatrix} \qquad P = \begin{bmatrix} P(1,1) & P(1,2) & \cdots & P(1,n) \\ P(2,1) & P(2,2) & & P(2,n) \\ \vdots & & \ddots & \vdots \\ P(n,1) & P(n,2) & \cdots & P(n,n) \end{bmatrix}$$

$$\pi_1 = \pi_0 P$$

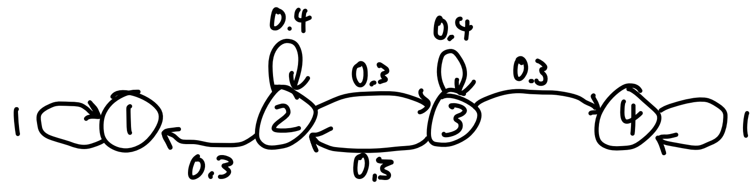
$$\pi_t = \pi_0 P^t$$

2 Can it be a Markov Chain?

(a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured. Given that the fly starts between positions 1 and m, model this process as a Markov Chain.



(b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4. Is the process Y_n a Markov chain?



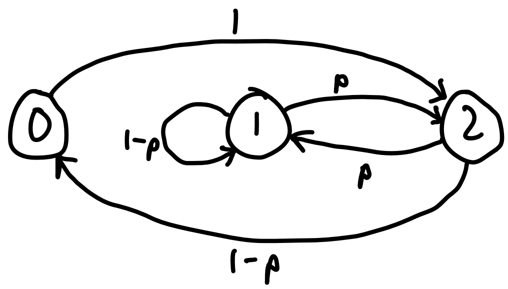
No, Markov property is violated.

(See solutions for counterexample.)

3 Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring exactly one umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.



$$\mathcal{X} = \{0, 1, 2\}$$

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} \end{array}$$

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

After 2 trips: $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} = \begin{bmatrix} 1-p & p & 0 \\ p(1-p) & (1-p)^2+p^2 & p(1-p) \\ 0 & p(1-p) & 1-p+p^2 \end{bmatrix}$

After n trips: P^n

Invariant distribution: $\pi = [\pi(0) \quad \pi(1) \quad \pi(2)]$

$\pi = \pi P$

$\pi(0) = (1-p)\pi(2)$

$\pi(1) = (1-p)\pi(1) + p\pi(2)$

$\pi(2) = \pi(0) + p\pi(1)$

$\pi(0) + \pi(1) + \pi(2) = 1$

$\pi(2) = (1-p)\pi(2) + p\pi(1)$

$\cancel{p}\pi(2) = \cancel{p}\pi(1)$

$(1-p)\pi(2) + \pi(2) + \pi(2) = 1$

$(3-p)\pi(2) = 1$

$\pi(2) = \frac{1}{3-p}$

$\pi(1) = \frac{1}{3-p}$

$\pi(0) = \frac{1-p}{3-p}$

Answer: $\frac{p(1-p)}{3-p}$