Common Continuous Random Variable Distributions

Distribution	PDF	CDF	$\mathbb{E}[X]$	$Var(X)$
Uniform$[a,b]$	$\begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential(λ)	$\begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$	$\begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal(μ, σ^2)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$	μ	σ^2

$\Phi(x)$ is the CDF of the standard normal $N(0,1)$ distribution: $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

Central Limit Theorem

The sum or average of many independent and identically distributed random variables will tend toward a normal distribution.

Given i.i.d. random variables $X_1, X_2, X_3, \ldots$ with finite mean μ and finite variance σ^2, define $S_n = \sum_{i=1}^{n} X_i$. Then the CDF of $\frac{S_n - n\mu}{\sigma\sqrt{n}}$ converges to $N(0,1)$ as $n \to \infty$.

1 Interesting Gaussians

(a) If $X \sim N(0, \sigma_X^2)$ and $Y \sim N(0, \sigma_Y^2)$ are independent, then what is $\mathbb{E}\left[(X+Y)^k\right]$ for any *odd* $k \in \mathbb{N}$?

$X + Y \sim \text{Normal}(0, \sigma_X^2 + \sigma_Y^2)$

$Z = X + Y$ is symmetric around 0.

$$E[(X+Y)^k] = E[Z^k] = \int_{-\infty}^{\infty} \underset{\uparrow \atop \text{odd}}{z^k}\, \underset{\uparrow \atop \text{even}}{f_Z(z)}\, dz = 0$$

(b) Let $f_{\mu,\sigma}(x)$ be the density of a $N(\mu, \sigma^2)$ random variable, and let X be distributed according to $\alpha f_{\mu_1,\sigma_1}(x) + (1-\alpha) f_{\mu_2,\sigma_2}(x)$ for some $\alpha \in [0,1]$. Compute $\mathbb{E}[X]$ and $\text{Var}(X)$. Is X normally distributed?

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x\left(\alpha f_{\mu_1\sigma_1}(x) + (1-\alpha) f_{\mu_2\sigma_2}(x)\right) dx \\ &= \alpha \int_{-\infty}^{\infty} x f_{\mu_1\sigma_1}(x)\, dx + (1-\alpha)\int_{-\infty}^{\infty} x f_{\mu_2\sigma_2}(x)\, dx \\ &= \alpha\mu_1 + (1-\alpha)\mu_2 \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2\left(\alpha f_{\mu_1\sigma_1}(x) + (1-\alpha) f_{\mu_2\sigma_2}(x)\right) dx \\ &= \alpha \int_{-\infty}^{\infty} x^2 f_{\mu_1\sigma_1}(x)\, dx + (1-\alpha)\int_{-\infty}^{\infty} x^2 f_{\mu_2\sigma_2}(x)\, dx \\ &= \alpha(\sigma_1^2 + \mu_1^2) + (1-\alpha)(\sigma_2^2 + \mu_2^2) \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \alpha(\sigma_1^2 + \mu_1^2) + (1-\alpha)(\sigma_2^2 + \mu_2^2) - (\alpha\mu_1 + (1-\alpha)\mu_2)^2 \end{aligned}$$

2 Binomial Concentration

Here, we will prove that the binomial distribution is *concentrated* about its mean as the number of trials tends to ∞. Suppose we have i.i.d. trials, each with a probability of success $1/2$. Let S_n be the number of successes in the first n trials (n is a positive integer), and define

$$Z_n := \frac{S_n - n/2}{\sqrt{n}/2}.$$

(a) What are the mean and variance of Z_n?

$S_n \sim \text{Binomial}(0, 1)$

$E[Z_n] = \frac{E[S_n] - n/2}{\sqrt{n}/2} = \frac{n/2 - n/2}{\sqrt{n}/2} = 0$

$Var(Z_n) = \frac{1}{n/4} Var(S_n - \frac{n}{2}) = \frac{4}{n} Var(S_n) = \frac{4}{n}\frac{n}{4} = 1$

(b) What is the distribution of Z_n as $n \to \infty$?

$Z_n \to \text{Normal}(0, 1)$

(c) Use the bound $\mathbb{P}[Z > z] \leq (\sqrt{2\pi}z)^{-1} e^{-z^2/2}$ when Z is a standard normal in order to approximately bound $\mathbb{P}[S_n/n > 1/2 + \delta]$, where $\delta > 0$.

$P[\frac{S_n}{n} > \frac{1}{2} + \delta] = P[\frac{S_n - n/2}{n} > \delta] = P[\frac{S_n - n/2}{\sqrt{n}/2} > 2\delta\sqrt{n}]$

$\approx P[Z > 2\delta\sqrt{n}] \leq \frac{1}{2\sigma\sqrt{n}\sqrt{2\pi}} e^{-\frac{(2\delta\sqrt{n})^2}{2}}$

3 Erasures, Bounds, and Probabilities

Alice is sending 1000 bits to Bob. The probability that a bit gets erased is p, and the erasure of each bit is independent of the others.

Alice is using a scheme that can tolerate up to one-fifth of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most 10^{-6}.

(a) Use Chebyshev's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.

X = number of erasures

$X \sim \text{Binomial}(1000, p)$

$E[X] = 1000p$

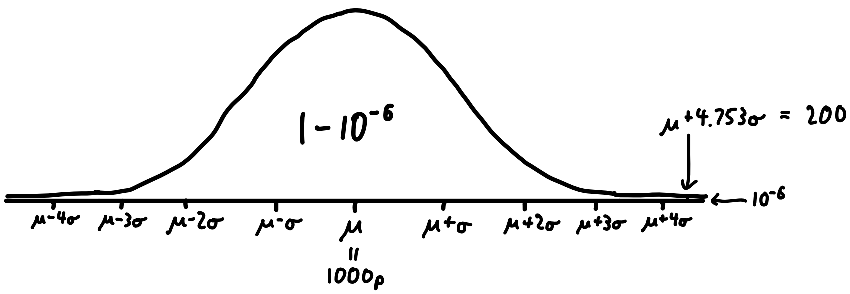
$\text{Var}(X) = 1000p(1-p)$

$$
\begin{aligned}
P[X \geq 200] &= P[X - 1000p \geq 200 - 1000p] \\
&\leq P[|X - 1000p| \geq 200 - 1000p] \\
&\leq \frac{1000p(1-p)}{(200-1000p)^2} \leq 10^{-6}
\end{aligned}
$$

$p \leq 3.998 \times 10^{-5}$

(b) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable (with suitable mean and variance). Use this to find an approximate bound for p such that the probability of a communications breakdown is at most 10^{-6}.

You may use that $\Phi^{-1}(1-10^{-6}) \approx 4.753$.



$$\mu + 4.753\sigma = 1000p + 4.753\sqrt{1000p(1-p)} \leq 200$$

$$p \leq 0.1468$$