- Fill out Midsemester Survey!

Markov's Inequality: For a nonnegative RV X,

$$P[X \geq c] \leq \frac{E[X]}{c}$$

Chebyshev's Inequality: $P[|X-\mu| \geq c] \leq \frac{Var(X)}{c^2}$

CS 70 Spring 2022 | Discrete Mathematics and Probability Theory Koushik Sen and Satish Rao | DIS 11B

1 Inequality Practice

(a) X is a random variable such that $X \geq -5$ and $\mathbb{E}[X] = -3$. Find an upper bound for the probability of X being greater than or equal to -1.

$Y = X + 5$

$P[X \geq -1] = P[Y \geq 4] \leq \frac{E[Y]}{4} = \frac{E[X+5]}{4} = \frac{1}{2}$

$P[X \geq -1] \leq \frac{1}{2}$

(b) Y is a random variable such that $Y \leq 10$ and $\mathbb{E}[Y] = 1$. Find an upper bound for the probability of Y being less than or equal to -1.

$Z = 10 - Y$

$E[Z] = 10 - 1 = 9$

$P[Y \leq -1] = P[Z \geq 11] \leq \frac{E[Z]}{11} = \frac{9}{11}$

(c) You roll a die 100 times. Let Z be the sum of the numbers that appear on the die throughout the 100 rolls. Compute $\text{Var}(Z)$. Then use Chebyshev's inequality to bound the probability of the sum Z being greater than 400 or less than 300.

$E[X_i] = \frac{7}{2}$

$Z = X_1 + X_2 + \ldots + X_{100}$

$Var(X_i) = \frac{35}{12}$

$Var(Z) = 100 \cdot \frac{35}{12}$

$P[|Z - 350| \geq 50] \leq \frac{100\left(\frac{35}{12}\right)}{50^2} = \frac{7}{60}$

2 Vegas

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip their coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Let X be the proportion of people whose coin flip results in heads. Find $\mathbb{E}[X]$.

$$X = \frac{1}{n}(X_1 + \dots + X_n) \qquad X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person flips heads} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_i] = p + (1-p)\frac{1}{2} = \frac{1}{2} + \frac{1}{2}p$$

$$E[X] = \frac{1}{n} \cdot n \cdot \frac{1}{2}(1+p)$$

$$= \frac{1}{2}(1+p)$$

(b) Given the results of your experiment, how should you estimate p? (*Hint:* Construct an unbiased estimator for p using part (a))

$$E[X] = \frac{1}{2}(1+p)$$

$$2E[X] = 1+p$$

$$p = 2E[X] - 1$$

$$\hat{p} = 2X - 1$$

(c) How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

$$Var(X) = Var\left(\frac{1}{n}(X_1 + \dots + X_n)\right)$$

$$= \frac{1}{n^2}\left(Var(X_1) + \dots + Var(X_n)\right) \qquad Var(X_i) \le \frac{1}{4}$$

$$\le \frac{1}{4n}$$

$$Var(\hat{p}) = Var(2X-1) = 4Var(X) \le \frac{1}{n}$$

$$P[|\hat{p} - p| \ge 0.05] \le \frac{Var(\hat{p})}{0.05^2} \le 1 - 0.95$$

$$Var(\hat{p}) \le 0.05^3$$

$$\frac{1}{n} \le 0.05^3$$

$$n \ge 8000$$

3 Working with the Law of Large Numbers

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

10 tosses

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

100 tosses

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

100 tosses

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

10 tosses